

ON CARTAN'S SECOND MAIN THEOREMS FOR HOLOMORPHIC CURVES ON M - PUNCTURED COMPLEX PLANES

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ABSTRACT. In this paper, we give some extension of fundamental theorems in Nevanlinna - Cartan theory for holomorphic curve on M - punctured complex planes. Detail, we prove some fundamental theorems for holomorphic curves on M - punctured complex plane $\Omega = \mathbb{C} \setminus \{c_1, \dots, c_M\}$ intersecting a finite set of fixed hyperplanes in $\mathbb{P}^n(\mathbb{C})$ and fixed hypersurfaces in general position on complex projective variety with the level of truncation, where $M \geq 2$ is an integer number and c_1, \dots, c_M are distinct complex numbers. Note that in here, the holomorphic curves may be contained the essential singularity points on complex plane (at $c_j, j = 1, \dots, M$) which have not been considered before in other references according to my understanding. As an application, we establish a result for uniqueness problem of holomorphic curve by inverse image of a hypersurface, it is improvement of some results before [8, 14] in this trend.

1. INTRODUCTION AND MAIN RESULTS

One of the main topic in studying the meromorphic functions is the Nevanlinna theory. In 1933, H. Cartan [2] generalized the Nevanlinna theory for holomorphic curve in projective space which is now called the Nevanlinna - Cartan theory. Since that time, this problem has been studied intensively by many authors. Nevanlinna - Cartan theory has found various applications in complex analysis and geometry complex such as uniqueness set theory, normal family theory, problem extension of holomorphic mapping and the property hyperbolic of algebraic variety.

In 2004, M. Ru [16] proved the second main theorem for hypersurface the counterpart of the result of Corvaja and Zannier [5] in approximation diophantine. In 2009, M. Ru [17] proved the second main theorem for holomorphic curves from \mathbb{C} into Algebraic variety on projective space. In 2010, Z. Chen, M. Ru and Q. Yan [3] gave an improvement of Ru's result [17] with the level of truncation. For meromorphic functions on Annuli in complex plane \mathbb{C} , in 2005, A. Y. Khrystyanyan and

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A. A. Kondratyuk [10, 11] showed some problems for the second main theorem and defect relation. In 2007, M. O. Hanyak and A. A. Kondratyuk [9] generalized the second main theorem of A. Y. Khrystianyn and A. A. Kondratyuk for M -punctured complex planes. In 2015, H. T. Phuong and N. V. Thin [13] have been generalized the results of A. Y. Khrystianyn and A. A. Kondratyuk for holomorphic curves on annuli intersecting a finite set of fixed hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ with ramification. In this paper, we will prove some fundamental theorems for holomorphic mappings from M -punctured complex planes to $\mathbb{P}^n(\mathbb{C})$ intersecting a finite set of hyperplanes which is an extension the result of Phuong and Thin [13]. Futhermore, we extend the second main theorem of M. Ru [17] for holomorphic curves from M -punctured complex planes into Algebraic variety in $\mathbb{P}^n(\mathbb{C})$ intersecting a finite set of hypersurfaces in general position.

First we remind some definitions in [9]. Let $c_j \in \mathbb{C}$, $j \in \{1, \dots, M\}$ be the set of distinct points, where $M \geq 2$ is a positive integer. Then, $\Omega = \mathbb{C} \setminus \cup_{j=1}^M \{c_j\}$ is called the M -punctured planes. Denotes $d = \frac{1}{2} \min\{|c_j - c_k| : j \neq k\}$ and $r_0 = 1/d + \max\{|c_j| : 1 \leq j \leq M\}$. It is easy to see that $1/r_0 < d$, $\overline{D}_{1/r_0}(c_j) \cap \overline{D}_{1/r_0}(c_k) = \emptyset$, $j \neq k$, and $\overline{D}_{1/r_0}(c_j) \subset D_{r_0}(0)$, $j \in \{1, 2, \dots, M\}$, where $D_r(c)$ denotes a disk of radius $r > 0$ centered at c . For an arbitrary $t \geq r_0$, we define

$$\Omega_t = D_t(0) \setminus \cup_{j=1}^M \overline{D}_{1/t}(c_j).$$

Using the above notation, we conclude that $\Omega_{r_0} \subset \Omega_r$, $r_0 < r \leq +\infty$.

Let $f = (f_0 : \dots : f_n) : \Omega \rightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic map where f_0, \dots, f_n are holomorphic functions and without common zeros in Ω . For $r_0 < r < +\infty$, the characteristic function $T_f(r)$ of f is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \|f(c_j + \frac{1}{r}e^{i\theta})\| d\theta,$$

where $\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\}$. The above definition is independent, up to an additive constant, of the choice of the reduced representation of f . Futhermore, when holomorphic curve f is holomorphic at c_j , $j = 1, \dots, M$, we have

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + O(1).$$

Therefore, $T_f(r)$ is Nevanlinna-Cartan characteristic function of holomorphic curve f on \mathbb{C} . Thus, our definition is an extension the definition of characteristic function for holomorphic curve on \mathbb{C} to $\Omega = \mathbb{C} \setminus \{c_1, \dots, c_M\}$. We add the

quantity

$$\frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \|f(c_j + \frac{1}{r}e^{i\theta})\| d\theta$$

in the original definition of $T_f(r)$ to control the growth of f in the neighborhood of the essential singularity points.

Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and

$$L(z_0, \dots, z_n) = \sum_{j=0}^n a_j z_j$$

be linear form defined H , where $a_j \in \mathbb{C}$, $j = 0, \dots, n$, are constants. Denote $a = (a_0, \dots, a_n)$ by the non-zero associated vector with H . And denote

$$(H, f) = (a, f) = \sum_{j=0}^n a_j f_j.$$

Under the assumption that $(a, f) \neq 0$, for $r_0 < r < +\infty$, the proximity function of f with respect to H is defined as

$$m_f(r, H) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|}{|(a, f)(re^{i\theta})|} d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a, f)(c_j + \frac{1}{r}e^{i\theta})|} d\theta,$$

where the above definition is independent, up to an additive constant, of the choice of the reduced representation of f .

We denote $n_f(r, H)$ by the number of zeros of (a, f) in $\overline{\Omega}_r$. The counting function of f is defined by

$$N_f(r, H) = \int_{r_0}^r \frac{n_f(t, H)}{t} dt.$$

Now let δ be a positive integer, we denote $n_f^\delta(r, H)$ by the numbers of zeros of (a, f) in $\overline{\Omega}_r$, where any zero of multiplicity greater than δ is “truncated” and counted as if it only had multiplicity δ . The truncated counting function of f is defined by

$$N_f^\delta(r, H) = \int_{r_0}^r \frac{n_f^\delta(t, H)}{t} dt.$$

Recall that hyperplanes H_1, \dots, H_q , $q > n$, in $\mathbb{P}^n(\mathbb{C})$ are said to be in general position if for any distinct $i_1, \dots, i_{n+1} \in \{1, \dots, q\}$,

$$\bigcap_{k=1}^{n+1} \text{supp}(H_{i_k}) = \emptyset,$$

this is equivalence to the $H_{i_1}, \dots, H_{i_{n+1}}$ being linearly independent.

In the case of hypersurface, we may define the proximity function, counting functions of holomorphic curve f similarly. Let D be a hypersurface in $\mathbb{P}^n(\mathbb{C})$ of degree d . Let Q be the homogeneous polynomial of degree d defining D . Under the assumption that $Q(f) \not\equiv 0$. Then, the proximity function $m_f(r, D)$ of f is defined by

$$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d}{|Q(f)(c_j + \frac{1}{r}e^{i\theta})|} d\theta,$$

where the above definition is independent, up to an additive constant, of the choice of the reduced representation of f . The next, we denote $n_f(r, D)$ by the number of zeros of $Q(f)$ in $\overline{\Omega}_r$. The counting function $N_f(r, D)$ of f is defined by

$$N_f(r, D) = \int_{r_0}^r \frac{n_f(t, D)}{t} dt.$$

Now let δ be a positive integer, we denote $n_f^\delta(r, D)$ by the numbers of zeros of $Q(f)$ in $\overline{\Omega}_r$, where any zero of multiplicity greater than δ is “truncated” and counted as if it only had multiplicity δ . The truncated counting function of f is defined by

$$N_f^\delta(r, D) = \int_{r_0}^r \frac{n_f^\delta(t, D)}{t} dt.$$

Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a smooth complex projective variety of dimension $n \geq 1$. Let D_1, \dots, D_q be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$, where $q > n$. The hypersurfaces D_1, \dots, D_q are said to be *in general position on V* if for every subset $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$, we have

$$V \cap \text{Supp} D_{i_0} \cap \dots \cap \text{Supp} D_{i_n} = \emptyset,$$

where $\text{Supp}(D)$ means the support of the divisor D . A map $f : \Omega \rightarrow V$ is said to be *algebraically nondegenerate* if the image of f is not contained in any proper subvarieties of V .

In this paper, a notation “ $\|$ ” in the inequality means that the inequality holds for $r \in (r_0, +\infty)$ outside a set of finite measure.

Our main results are

Theorem 1. *Let D be a hypersurfaces in $\mathbb{P}^n(\mathbb{C})$ defining the homogeneous polynomial Q with degree d and $f = (f_0 : \dots : f_n) : \Omega \rightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained D . Then we have for any $r_0 < r < +\infty$,*

$$dT_f(r) = m_f(r, D) + N_f(r, D) + O(1),$$

where $O(1)$ is a constant independent of r .

Note that, when $d = 1$, we get the corollary as following:

Corollary 1. *Let H be a hyperplane in $\mathbb{P}^n(\mathbb{C})$ and $f = (f_0 : \cdots : f_n) : \Omega \rightarrow \mathbb{P}^n(\mathbb{C})$ be a holomorphic curve whose image is not contained H . Then we have for any $r_0 < r < +\infty$,*

$$T_f(r) = m_f(r, H) + N_f(r, H) + O(1),$$

where $O(1)$ is a constant independent of r .

Theorem 2. *Let $f = (f_0 : \cdots : f_n) : \Omega \rightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and H_1, \dots, H_q be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have for any $r_0 < r < +\infty$,*

$$\| \quad (q - n - 1)T_f(r) \leq \sum_{l=1}^q N_f^n(r, H_l) + O(\log r + \log T_f(r)).$$

Theorem 3. *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a complex projective variety of dimension $n \geq 1$. Let D_1, \dots, D_q be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree d_j , located in general position on V . Let d be the least common multiple of the d_i , $i = 1, \dots, q$. Let $f = (f_0 : \cdots : f_N) : \Omega \rightarrow V$ be an algebraically non-degenerate holomorphic map. Let $\varepsilon > 0$ and*

$$\alpha \geq \frac{n^n d^{n^2+n} (19nI(\varepsilon^{-1}))^n (\deg V)^{n+1}}{n!},$$

where $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for a positive real number x . Then

$$\| \quad (q(1 - \varepsilon/3) - (n + 1) - \varepsilon/3)T_f(r) \leq \sum_{l=1}^q d_l^{-1} N_f^\alpha(r, Q_l) + O(\log r + \log T_f(r)).$$

When f is holomorphic at $c_j, j = 1, \dots, M$, we get following result of Ru et. al [3]. Thus Theorem 3 is an extension of Ru et. al [3] and Ru [17] for holomorphic curve with essential singularity points.

Theorem 4. *Let $V \subset \mathbb{P}^N(\mathbb{C})$ be a complex projective variety of dimension $n \geq 1$. Let D_1, \dots, D_q be hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ of degree d_j , located in general position on V . Let d be the least common multiple of the d_i , $i = 1, \dots, q$. Let $f = (f_0 : \cdots : f_N) : \mathbb{C} \rightarrow V$ be an algebraically non-degenerate holomorphic map. Let $\varepsilon > 0$ and*

$$\alpha \geq \frac{n^n d^{n^2+n} (19nI(\varepsilon^{-1}))^n (\deg V)^{n+1}}{n!},$$

where $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for a positive real number x . Then

$$(q(1 - \varepsilon/3) - (n + 1) - \varepsilon/3)T_f(r) \leq \sum_{l=1}^q d_l^{-1} N_f^\alpha(r, Q_l) + O(\log r + \log T_f(r)),$$

holds for all $r \in (0, +\infty)$ outside a set of finite measure.

Theorem 1 and Corollary 1 is first main theorem, and Theorem 2 is second main theorem for holomorphic curves from M -punctured Ω to $\mathbb{P}^n(\mathbb{C})$ intersecting a collection of fixed hyperplanes in general position with truncated counting functions. Theorem 3 is second main theorem for holomorphic curves from M -punctured Ω to V intersecting a collection of fixed hypersurfaces in general position with the level of truncation. When one applies inequalities of second main theorem type, it is often crucial to the application to have the inequality with truncated counting functions. For example, all existing constructions of unique range sets depend on a second main theorem with truncated counting functions.

Theorem 5. *Let $f : \Omega \rightarrow \mathbb{P}^N(\mathbb{C})$ be an algebraically nondegenerate holomorphic curve. Let d and n be two integers with $n > N(d + N + 1)$. Let $\mathcal{H}_i = \{z \in \mathbb{P}^N(\mathbb{C}), \mathcal{H}_i(z) = 0\}, 0 \leq i \leq N$, be hyperplanes in $\mathbb{P}^N(\mathbb{C})$. Let $D_i = \{z \in \mathbb{P}^N(\mathbb{C}), Q_i(z) = 0\}, 0 \leq i \leq N$, be hypersurfaces of degree d such that the hypersurfaces $\{\mathcal{H}_0^n Q_0 = 0\}, \dots, \{\mathcal{H}_N^n Q_N = 0\}$ are in general position in $\mathbb{P}^N(\mathbb{C})$. Let $D = \{z \in \mathbb{P}^N(\mathbb{C}), \sum_{i=0}^N \mathcal{H}_i^n Q_i = 0\}$. Then*

$$\begin{aligned} \|(n - (d + N + 1)N)T_f(r) + \sum_{i=0}^N (N_f(r, D_i) - N_f^N(r, D_i)) \\ \leq N_f^N(r, D) + o(T_f(r)). \end{aligned}$$

We give a hypersurfaces satisfying Theorem 5.

Example 6. Let $D_i = \{z = (x_0 : \dots : x_N) \in \mathbb{P}^N(\mathbb{C}), x_i^d = 0\}, 0 \leq i \leq N$, be hypersurfaces of degree d . Let $\mathcal{H}_i = \{z = (x_0 : \dots : x_N) \in \mathbb{P}^N(\mathbb{C}), \sum_{t=0}^i x_t = 0\}$. We see that the hypersurfaces $\{(\sum_{t=0}^i x_t)^n x_i^d = 0\}, 0 \leq i \leq N$, are in general position in $\mathbb{P}^N(\mathbb{C})$. Then

$$D = \{z \in \mathbb{P}^N(\mathbb{C}), \sum_{i=0}^N (\sum_{t=0}^i x_t)^n x_i^d = 0\}$$

satisfies the Theorem 5 with $n > N(d + N + 1)$.

As an application of Theorem 5, we prove the uniqueness theorem for holomorphic curves on M -punctured complex plane by inverse image of a Fermat hypersurface.

Theorem 7. *Let $f, g : \Omega \rightarrow \mathbb{P}^N(\mathbb{C})$ be two algebraically nondegenerate holomorphic curves, and n be a integer with $n > N(d + N + 3)$. Let D be a hypersurface as in Theorem 5. Suppose that $f(z) = g(z)$ on $f^{-1}(D) \cup g^{-1}(D)$, then $f \equiv g$.*

When f and g are holomorphic at $c_j, j = 1, \dots, M$, we obtain the corollary as following:

Corollary 2. *Let $f, g : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ be two algebraically nondegenerate holomorphic curves, and n be a integer with $n > N(d + N + 3)$. Let D be a hypersurface as in Theorem 5. Suppose that $f(z) = g(z)$ on $f^{-1}(D) \cup g^{-1}(D)$, then $f \equiv g$.*

We reduce the number hypersurfaces in before results. The authors [6, 8, 14] studied the uniqueness problem with a number lager hypersurfaces. Here, we only need a hypersurface.

2. SOME PRELIMINARIES IN NEVANLINNA THEORY FOR MEROMORPHIC FUNCTIONS

In order to prove theorems, we need the following lemmas. Let f be a meromorphic function on M -punctured Ω and $r \in (r_0, +\infty)$, we recall that

$$\begin{aligned} m_0(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log^+ |f(c_j + \frac{1}{r}e^{i\theta})| d\theta \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(r_0e^{i\theta})| d\theta - \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log^+ |f(c_j + \frac{1}{r_0}e^{i\theta})| d\theta. \end{aligned}$$

We denote $n_0(r, f)$ by the numbers of its poles in $\bar{\Omega}_r$. The counting function $N_0(r, f)$ of f is defined by

$$N_0(r, f) = \frac{1}{2\pi} \int_{r_0}^r \frac{n_0(t, f)}{t} dt.$$

The function

$$T_0(r, f) = N_0(r, f) + m_0(r, f)$$

is called *the Nevanlinna characteristic of f* .

Lemma 1. [9] (*Jensen's Theorem for M -punctured planes*) *Let f be a non-constant meromorphic function in an M -punctured plane Ω not identically equal to zero and let $r_0 < r < +\infty$. Then*

$$\begin{aligned} N_0\left(r, \frac{1}{f}\right) - N_0(r, f) &= \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\theta})| d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log |f(c_j + \frac{1}{r}e^{i\theta})| d\theta \\ &\quad - \frac{1}{2\pi} \int_0^{2\pi} \log |f(r_0e^{i\theta})| d\theta - \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log |f(c_j + \frac{1}{r_0}e^{i\theta})| d\theta. \end{aligned}$$

Lemma 2. [9] *Let f be a non-constant meromorphic function on Ω . Then, we have the equality*

$$\| m_0(r, \frac{f'}{f}) = O(\log r + \log T_0(r, f)),$$

holds for all $r \in (r_0, +\infty)$ outside a set of finite measure.

Let $X \subset \mathbb{P}^N(\mathbb{C})$ be a projective variety of dimension n and degree Δ . Let I_X be the prime ideal in $\mathbb{C}[x_0, \dots, x_N]$ defining X . Denote by $\mathbb{C}[x_0, \dots, x_N]_m$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_N]$ of degree m (including 0). Put $I_X(m) := \mathbb{C}[x_0, \dots, x_N]_m \cap I_X$. The Hilbert function H_X of X is defined by

$$H_X(m) := \dim \mathbb{C}[x_0, \dots, x_N]_m / I_X(m).$$

For each tuple $c = (c_0, \dots, c_N) \in \mathbb{R}_{\geq 0}^{N+1}$ and $m \in \mathbb{N}$, we define the m -th Hilbert weight $S_X(m, c)$ of X with respect to c by

$$S_X(m, c) := \max \sum_{i=1}^{H_X(m)} I_i \cdot c,$$

where $I_i = (I_{i0}, \dots, I_{iN}) \in \mathbb{N}_0^{N+1}$ and the maximum is taken over all sets $\{x^{I_i} = x_0^{I_{i0}} \dots x_N^{I_{iN}}\}$ whose residue classes modulo $I_X(m)$ form a basis of the vector space $\mathbb{C}[x_0, \dots, x_N]_m / I_X(m)$.

Lemma 3. [17] *Let $X \subset \mathbb{P}^N(\mathbb{C})$ be an algebraic variety of dimension n and degree Δ . Let $m > \Delta$ be an integer and let $c = (c_0, \dots, c_N) \in \mathbb{R}_{\geq 0}^{N+1}$. Then*

$$\frac{1}{m H_X(m)} S_X(m, c) \geq \frac{1}{(n+1)\Delta} e_X(c) - \frac{(2n+1)\Delta}{m} \max_{0 \leq i \leq N} c_i.$$

Lemma 4. [17] *Let Y be a subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$ of dimension n and degree Δ . Let $c = (c_1, \dots, c_q)$ be a tupe of positive reals. Let $\{i_0, \dots, i_n\}$ be a subset of $\{1, \dots, q\}$ such that $\{y_{i_0} = \dots = y_{i_n} = 0\} \cap Y = \emptyset$. Then*

$$e_Y(c) \geq (c_{i_0} + \dots c_{i_n})\Delta.$$

3. PROOF OF THEOREMS

Proof of Theorem 1. The first, we note that $N_0(r, Q(f)) = 0$. By the definitions of $T_f(r)$, $N_f(r, H)$, $m_f(r, H)$ and apply to Lemma 1 for $Q(f) \not\equiv 0$, we have

$$\begin{aligned} N_f(r, D) &= N_0(r, \frac{1}{Q(f)}) \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log |Q(f)(re^{i\theta})| d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log |Q(f)(c_j + \frac{1}{r}e^{i\theta})| d\theta + O(1). \end{aligned}$$

Hence, we get

$$\begin{aligned}
& N_f(r, D) + m_f(r, D) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d}{|Q(f)(c_j + \frac{1}{r}e^{i\theta})|} d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} \log |Q(f)(re^{i\theta})| d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log |Q(f)(c_j + \frac{1}{r}e^{i\theta})| d\theta + O(1) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\|^d d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \|f(c_j + \frac{1}{r}e^{i\theta})\|^d d\theta + O(1) \\
&= dT_f(r) + O(1).
\end{aligned}$$

This is conclusion of Theorem 1. \square

Proof of Theorem 2. To prove the Theorem 2, we need some lemmas. First we recall the property of Wronskian. Let $f = (f_0 : \dots : f_n) : \Omega \rightarrow \mathbb{P}^n(\mathbb{C})$ be holomorphic curves, the determinant of Wronskian of f is defined by

$$W = W(f) = W(f_0, \dots, f_n) = \begin{vmatrix} f_0(z) & f_1(z) & \cdot & f_n(z) \\ f_0'(z) & f_1'(z) & \cdot & f_n'(z) \\ \vdots & \vdots & \ddots & \vdots \\ f_0^{(n)}(z) & f_1^{(n)}(z) & \cdot & f_n^{(n)}(z) \end{vmatrix}.$$

We denote by $N_W(r, 0)$ the counting function of zeros of $W(f_0, \dots, f_n)$ in $\overline{\Omega}_r$, namely

$$N_W(r, 0) = N_0(r, \frac{1}{W}) + O(1).$$

Let L_0, \dots, L_n are linearly independent forms of z_0, \dots, z_n . For $j = 0, \dots, n$, set

$$F_j(z) := L_j(f(z)).$$

By the property of Wronskian there exists a constant $C \neq 0$ such that

$$|W(F_0, \dots, F_n)| = C|W(f_0, \dots, f_n)|.$$

Lemma 5. *Let $f = (f_0 : \dots : f_n) : \Omega \rightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and H_1, \dots, H_q be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Then we have the inequality*

$$\begin{aligned}
& \left\| \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \sum_{j=1}^M \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} \right. \\
& \quad \left. \leq (n+1)T_f(r) - N_W(r, 0) + O(\log r + \log T_f(r)). \right.
\end{aligned}$$

Here the maximum is taken over all subsets K of $\{1, \dots, q\}$ such that $a_l, l \in K$, are linearly independent.

First, we prove

$$(3.1) \quad \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \frac{1}{2\pi} \int_0^{2\pi} \log |W(f)(re^{i\theta})| d\theta \\ \leq (n+1) \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + O(\log r + \log T_f(r)),$$

holds for any $r \in (r_0, +\infty)$. Let $K \subset \{1, \dots, q\}$ such that $a_l, l \in K$, are linearly independent. Without loss of generality, we may assume that $q \geq n+1$ and $\#K = n+1$. Let \mathcal{T} is the set of all injective maps $\mu : \{0, 1, \dots, n\} \rightarrow \{1, \dots, q\}$. Then we have

$$\begin{aligned} & \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_j, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \max_{\mu \in \mathcal{T}} \sum_{l=0}^n \log \frac{\|f(re^{i\theta})\|}{|(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log \left\{ \max_{\mu \in \mathcal{T}} \frac{\|f(re^{i\theta})\|^{n+1}}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1) \\ &\leq \int_0^{2\pi} \log \sum_{\mu \in \mathcal{T}} \frac{\|f(re^{i\theta})\|^{n+1}}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1) \\ &= \int_0^{2\pi} \log \sum_{\mu \in \mathcal{T}} \frac{|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &\quad + \int_0^{2\pi} \log \sum_{\mu \in \mathcal{T}} \frac{\|f(re^{i\theta})\|^{n+1}}{|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1). \end{aligned}$$

By the property of Wronskian, we see that

$$|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))| = C|W(f_0, \dots, f_n)|,$$

where $C \neq 0$ is a constant. So we obtain

$$\begin{aligned}
 (3.2) \quad & \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
 & \leq \int_0^{2\pi} \log \sum_{\mu \in T} \frac{|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
 & \quad + \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^{n+1}}{|W(f_0, \dots, f_n)(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1).
 \end{aligned}$$

Take

$$g_{\mu(j)} = \frac{(a_{\mu(j)}, f)}{(a_{\mu(0)}, f)}, j = 1, \dots, n.$$

From property of Wronskian (see [12], Proposition 1.4.3), we have

$$\begin{aligned}
 (3.3) \quad & \frac{W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))}{\prod_{j=0}^n (a_{\mu(j)}, f)} = \frac{W(1, \frac{(a_{\mu(1)}, f)}{(a_{\mu(0)}, f)}, \dots, \frac{(a_{\mu(n)}, f)}{(a_{\mu(0)}, f)})}{\frac{(a_{\mu(1)}, f)}{(a_{\mu(0)}, f)} \cdots \frac{(a_{\mu(n)}, f)}{(a_{\mu(0)}, f)}} \\
 & = \begin{vmatrix} 1 & 1 & \dots & 1 \\ 0 & \frac{g'_{\mu(1)}}{g_{\mu(1)}} & \dots & \frac{g'_{\mu(n)}}{g_{\mu(n)}} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{g_{\mu(1)}^{(n)}}{g_{\mu(1)}} & \dots & \frac{g_{\mu(n)}^{(n)}}{g_{\mu(n)}} \end{vmatrix}.
 \end{aligned}$$

We see

$$\begin{aligned}
 (3.4) \quad & \parallel m\left(r, \frac{g_{\mu(j)}^{(k)}}{g_{\mu(j)}}\right) \leq m_0\left(r, \frac{g_{\mu(j)}^{(k)}}{g_{\mu(j)}}\right) = m_0\left(r, \frac{g_{\mu(j)}^{(k)}}{g_{\mu(j)}^{(k-1)}} \frac{g_{\mu(j)}^{(k-1)}}{g_{\mu(j)}^{(k-2)}} \cdots \frac{g'_{\mu(j)}}{g_{\mu(j)}}\right) \\
 & \leq \sum_{\nu=1}^k m_0\left(r, \frac{g_{\mu(j)}^{(\nu)}}{g_{\mu(j)}^{(\nu-1)}}\right).
 \end{aligned}$$

By Lemma 2, we have

$$(3.5) \quad m_0\left(r, \frac{g'_{\mu(j)}}{g_{\mu(j)}}\right) = O(\log r + \log T_0(r, g_{\mu(j)})).$$

From the definition of $T_0(r, g'_{\mu(j)})$, $N_0(r, g'_{\mu(j)})$ and (3.5), we obtain

(3.6)

$$\begin{aligned}
T_0(r, g'_{\mu(j)}) &= m_0(r, g'_{\mu(j)}) + N_0(r, g'_{\mu(j)}) \\
&= m_0\left(r, \frac{g'_{\mu(j)}}{g_{\mu(j)}} \cdot g_{\mu(j)}\right) + N_0(r, g'_{\mu(j)}) \\
&\leq m_0(r, g_{\mu(j)}) + N_0(r, g_{\mu(j)}) + \overline{N}_0(r, g_{\mu(j)}) + O(\log r + \log T_0(r, g_{\mu(j)})) \\
&= 2T_0(r, g_{\mu(j)}) + O(\log r + \log T_0(r, g_{\mu(j)})).
\end{aligned}$$

Similarly, again using Lemma 2 and (3.6), we have

(3.7)

$$\begin{aligned}
T_0(r, g''_{\mu(j)}) &= m_0(r, g''_{\mu(j)}) + N_0(r, g''_{\mu(j)}) \\
&= m_0\left(r, \frac{g''_{\mu(j)}}{g'_{\mu(j)}} \cdot g'_{\mu(j)}\right) + N_0(r, g''_{\mu(j)}) \\
&\leq m_0(r, g'_{\mu(j)}) + N_0(r, g_{\mu(j)}) + 2\overline{N}_0(r, g_{\mu(j)}) + O(\log r + \log T_0(r, g'_{\mu(j)})) \\
&= 3T_0(r, g_{\mu(j)}) + O(\log r + \log T_0(r, g_{\mu(j)})).
\end{aligned}$$

By argument as (3.7) and using inductive method, we obtain that the inequality

$$(3.8) \quad T_0(r, g_{\mu(j)}^{(\nu)}) \leq (\nu + 1)T_0(r, g_{\mu(j)}) + O(\log r + \log T_0(r, g_{\mu(j)}))$$

holds for all $\nu \in \mathbb{N}^*$. Furthermore, by Lemma 2, we also have the equality

$$(3.9) \quad m_0\left(r, \frac{g_{\mu(j)}^{(\nu+1)}}{g_{\mu(j)}^{(\nu)}}\right) = O(\log r + \log T_0(r, g_{\mu(j)}^{(\nu)})),$$

holds for all $\nu \in \mathbb{N}$. Combining (3.4), (3.8) and (3.9), we get for any $k \in \{1, \dots, n\}$ and $j \in \{1, \dots, n\}$,

$$(3.10) \quad \parallel \quad m\left(r, \frac{g_{\mu(j)}^{(k)}}{g_{\mu(j)}}\right) \leq O(\log r + \log T_0(r, g_{\mu(j)})).$$

By the definition of $T_0(r, g_{\mu(j)}), T_f(r)$, we have

$$\begin{aligned}
(3.11) \quad T_0(r, g_{\mu(j)}) &= m_0(r, g_{\mu(j)}) + N_0(r, g_{\mu(j)}) \\
&= \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{(a_{\mu(j)}, f)}{(a_{\mu(0)}, f)} \right| d\theta + \frac{1}{2\pi} \sum_{j=1}^m \int_0^{2\pi} \log^+ \left| \frac{(a_{\mu(j)}, f)}{(a_{\mu(0)}, f)} (c_j + \frac{1}{r} e^{i\theta}) \right| d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} \log |(a_{\mu(0)}, f)| d\theta + \frac{1}{2\pi} \sum_{j=1}^m \int_0^{2\pi} \log |(a_{\mu(0)}, f) (c_j + \frac{1}{r} e^{i\theta})| d\theta + O(1) \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \log \left| \frac{(a_{\mu(j)}, f) + (a_{\mu(0)}, f)}{(a_{\mu(0)}, f)} \right| d\theta \\
&\quad + \frac{1}{2\pi} \sum_{j=1}^m \int_0^{2\pi} \log \left| \frac{(a_{\mu(j)}, f) + (a_{\mu(0)}, f)}{(a_{\mu(0)}, f)} (c_j + \frac{1}{r} e^{i\theta}) \right| d\theta \\
&\quad + \frac{1}{2\pi} \int_0^{2\pi} \log |(a_{\mu(0)}, f)| d\theta + \frac{1}{2\pi} \sum_{j=1}^m \int_0^{2\pi} \log |(a_{\mu(0)}, f) (c_j + \frac{1}{r} e^{i\theta})| d\theta + O(1) \\
&\leq T_f(r) + O(1).
\end{aligned}$$

Hence for any $\mu \in \mathcal{T}$, from (3.3), (3.10) and (3.11), we have

$$\| \int_0^{2\pi} \log^+ \frac{|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \leq O(\log r + \log T_f(r)).$$

This implies

$$\begin{aligned}
(3.12) \quad &\| \int_0^{2\pi} \log \sum_{\mu \in \mathcal{T}} \frac{|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&\leq \int_0^{2\pi} \log^+ \sum_{\mu \in \mathcal{T}} \frac{|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
&\leq \sum_{\mu \in \mathcal{T}} \int_0^{2\pi} \log^+ \frac{|W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f))(re^{i\theta})|}{\prod_{l=0}^n |(a_{\mu(l)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + O(1) \\
&\leq O(\log r + \log T_f(r)).
\end{aligned}$$

We may obtain the inequality (3.1) from (3.2) and (3.12). Similarly, we get (3.13)

$$\begin{aligned} & \left\| \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} + \frac{1}{2\pi} \int_0^{2\pi} \log |W(f)(c_j + \frac{1}{r}e^{i\theta})| d\theta \right. \\ & \quad \left. \leq (n+1) \frac{1}{2\pi} \int_0^{2\pi} \log \|f(c_j + \frac{1}{r}e^{i\theta})\| d\theta + O(\log r + \log T_f(r)) \right\| \end{aligned}$$

holds for any $r \in (r_0, +\infty)$ and for all $j = 1, \dots, M$. Combining (3.1) and (3.13) we obtain

$$\begin{aligned} & \left\| \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \sum_{j=1}^M \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} \right. \\ & \quad \leq (n+1) \left(\frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \|f(c_j + \frac{1}{r}e^{i\theta})\| d\theta \right) \\ & \quad - \frac{1}{2\pi} \left(\int_0^{2\pi} \log |W(f)(re^{i\theta})| d\theta + \sum_{j=1}^M \int_0^{2\pi} \log |W(f)(c_j + \frac{1}{r}e^{i\theta})| d\theta \right) \\ & \quad \left. + O(\log r + \log T_f(r)) \right\|. \end{aligned}$$

Since

$$N_W(r, 0) = \frac{1}{2\pi} \int_0^{2\pi} \log |W(f)(re^{i\theta})| d\theta + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log |W(f)(c_j + \frac{1}{r}e^{i\theta})| d\theta + O(1),$$

we have the conclusion of this lemma. \square

Lemma 6. *Let $f = (f_0 : \dots : f_n) : \Omega \longrightarrow \mathbb{P}^n(\mathbb{C})$ be a linearly non-degenerate holomorphic curve and H_1, \dots, H_q be hyperplanes in $\mathbb{P}^n(\mathbb{C})$ in general position. Let a_j be the vector associated with H_j for $j = 1, \dots, q$. Then*

$$\begin{aligned} \sum_{l=1}^q m_f(r, H_l) & \leq \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\ & \quad + \sum_{j=1}^M \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1). \end{aligned}$$

Proof. Let $a_l = (a_0^{(l)}, \dots, a_n^{(l)})$ be the associated vector of H_l , $1 \leq l \leq q$, and let \mathcal{T} be the set of all injective maps $\mu : \{0, 1, \dots, n\} \longrightarrow \{1, \dots, q\}$. By hypothesis H_1, \dots, H_q are in general position for any $\mu \in \mathcal{T}$, then the vectors $a_{\mu(0)}, \dots, a_{\mu(n)}$ are linearly independent.

Let $\mu \in T$, we have

$$(3.14) \quad (f, a_{\mu(t)}) = a_0^{\mu(t)} f_0 + \cdots + a_n^{\mu(t)} f_n, \quad t = 0, 1, \dots, n.$$

Solve the system of linear equations (3.14), we get

$$f_t = b_0^{\mu(t)}(a_0^{\mu(t)}, f) + \cdots + b_n^{\mu(t)}(a_n^{\mu(t)}, f), \quad t = 0, 1, \dots, n,$$

where $\left(b_j^{\mu(t)}\right)_{t,j=0}^n$ is the inverse matrix of $\left(a_j^{\mu(t)}\right)_{t,j=0}^n$. So there is a constant C_μ satisfying

$$\|f(z)\| \leq C_\mu \max_{0 \leq t \leq n} |(a_{\mu(t)}, f)(z)|.$$

Set $C = \max_{\mu \in \mathcal{T}} C_\mu$. Then for any $\mu \in \mathcal{T}$, we have

$$\|f(z)\| \leq C \max_{0 \leq t \leq n} |(a_{\mu(t)}, f)(z)|.$$

For any $z \in \Omega$, there exists the mapping $\mu \in \mathcal{T}$ such that

$$0 < |(a_{\mu(0)}, f)(z)| \leq |(a_{\mu(1)}, f)(z)| \leq \cdots \leq |(a_{\mu(n)}, f)(z)| \leq |(a_l, f)(z)|,$$

for $l \notin \{\mu(0), \dots, \mu(n)\}$. Hence

$$\prod_{l=1}^q \frac{\|f(z)\|}{|(a_l, f)(z)|} \leq C^{q-n-1} \max_{\mu \in \mathcal{T}} \prod_{t=0}^n \frac{\|f(z)\|}{|(a_{\mu(t)}, f)(z)|}.$$

We have

$$\begin{aligned} & \sum_{l=1}^q m_f(r, H_l) \\ &= \sum_{l=1}^q \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \sum_{l=1}^q \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \prod_{l=1}^q \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \prod_{l=1}^q \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi}. \end{aligned}$$

This implies

$$\begin{aligned}
& \sum_{l=1}^q m_f(r, H_l) \\
& \leq \frac{1}{2\pi} \int_0^{2\pi} \log \max_{\mu \in \mathcal{T}} \prod_{t=0}^n \frac{\|f(re^{i\theta})\|}{|(a_{\mu(t)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
& \quad + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \log \max_{\mu \in \mathcal{T}} \prod_{t=0}^n \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_{\mu(t)}, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1) \\
& = \frac{1}{2\pi} \int_0^{2\pi} \max_{\mu \in T} \log \prod_{t=0}^n \frac{\|f(re^{i\theta})\|}{|(a_{\mu(t)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
& \quad + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \max_{\mu \in T} \log \prod_{t=0}^n \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_{\mu(t)}, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1) \\
& = \frac{1}{2\pi} \int_0^{2\pi} \max_{\mu \in T} \sum_{t=0}^n \log \frac{\|f(re^{i\theta})\|}{|(a_{\mu(t)}, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
& \quad + \frac{1}{2\pi} \sum_{j=1}^M \int_0^{2\pi} \max_{\mu \in T} \sum_{t=0}^n \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_{\mu(t)}, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1).
\end{aligned}$$

So we obtain

$$\begin{aligned}
\sum_{l=1}^q m_f(r, H_l) & \leq \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\
& \quad + \sum_{j=1}^M \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1).
\end{aligned}$$

This is conclusion of the Lemma 3. □

Proof of Theorem 2. By Lemma 5 and Lemma 6, we obtain

(3.15)

$$\begin{aligned} \parallel \sum_{l=1}^q m_f(r, H_l) &\leq \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &\quad + \sum_{j=1}^M \int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|}{|(a_l, f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} + O(1) \\ &\leq (n+1)T_f(r) - N_W(r, 0) + O(\log r + \log T_f(r)). \end{aligned}$$

By Theorem 1, we get that

$$T_f(r) = N_f(r, H_j) + m_f(r, H_j) + O(1)$$

for any $j \in \{1, \dots, q\}$. So from (3.15), we have

$$(3.16) \quad \parallel (q-n-1)T_f(r) \leq \sum_{l=1}^q N_f(r, H_l) - N_W(r, 0) + O(\log r + \log T_f(r)).$$

For $z_0 \in \overline{\Omega}_r$, we may assume that (a_l, f) vanishes at z_0 for $1 \leq l \leq q_1$, (a_l, f) does not vanish at z_0 for $l > q_1$. Hence, there exists a integer k_l and nowhere vanishing holomorphic function g_l in neighborhood U of z such that

$$(a_l, f)(z) = (z - z_0)^{k_l} g_l(z), \text{ for } l = 1, \dots, q,$$

here $k_l = 0$ for $q_1 < l \leq q$. We may assume that $k_l \geq n$ for $1 \leq l \leq q_0$, and $1 \leq k_l < n$ for $q_0 < l \leq q_1$. By property of the Wronskian, we have

$$W(f) = C.W((a_{\mu(0)}, f), \dots, (a_{\mu(n)}, f)) = \prod_{l=1}^{q_0} (z - z_0)^{k_l - n} h(z),$$

where $h(z)$ is holomorphic function on U . Then $W(f)$ is vanishes at z_0 with order at least

$$\sum_{l=1}^{q_0} (k_l - n) = \sum_{l=1}^{q_0} k_l - q_0 n.$$

By the definition of $N_f(r, H)$, $N_W(r, 0)$ and $N_f^n(r, H)$, we have

$$\begin{aligned} \sum_{l=1}^q N_f(r, H_j) - N_W(r, 0) &= \sum_{l=1}^q N_f(r, H_l) - N_0(r, \frac{1}{W}) + O(1) \\ &\leq \sum_{l=1}^q N_f^n(r, H_l) + O(1). \end{aligned}$$

So from (3.16), we have

$$\| (q - n - 1)T_f(r) \leq \sum_{l=1}^q N_f^n(r, H_l) + O(\log r + \log T_f(r)).$$

The proof of Theorem 2 is completed. \square

Proof of Theorem 3. Let D_1, \dots, D_q be the hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ which are located in general position on V . Let $Q_l, 1 \leq l \leq q$ be the homogeneous polynomials in $\mathbb{C}[x_0, \dots, x_N]$ of degree d_l defining on D_l . We can replace Q_l by Q_l^{d/d_l} , where d is the l.c.m of $d_l, l = 1, \dots, q$, we may assume that Q_1, \dots, Q_q have the same degree of d .

Given $z \in \Omega$ there exists a renumbering $\{i_0, \dots, i_n\}$ of the indices $\{1, \dots, q\}$ such that

$$(3.17) \quad 0 < |Q_{i_0} \circ (f(z))| \leq |Q_{i_2} \circ (f(z))| \leq \dots \leq |Q_{i_n} \circ (f(z))| \leq \min_{l \notin \{i_0, \dots, i_n\}} |Q_l \circ (f(z))|.$$

Suppose that P_1, \dots, P_s is a base of algebraic variety V . From the hypothesis, D_1, \dots, D_q are hypersurfaces in $\mathbb{P}^N(\mathbb{C})$ which are located in general position on V , we have for every subset $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$,

$$V \cap \text{Supp} D_{i_0} \cap \dots \cap \text{Supp} D_{i_n} = \emptyset.$$

This implies

$$P_1 \cap \dots \cap P_s \cap \text{Supp} D_{i_0} \cap \dots \cap \text{Supp} D_{i_n} = \emptyset.$$

Thus by Hilberts Nullstellensatz [19], for any integer $k, 0 \leq k \leq N$, there is an integer $m_k > \{d, \max_{t=1}^s \{\deg P_t\}\}$ such that

$$x_k^{m_k} = \sum_{j=0}^n b_{k_j}(x_0, \dots, x_N) Q_{i_j}(x_0, \dots, x_N) + \sum_{t=1}^s b_t(x_0, \dots, x_N) P_t(x_0, \dots, x_N),$$

where b_{k_j} are homogeneous forms with coefficients in \mathbb{C} of degree $m_k - d$ and b_t are homogeneous forms with coefficients in \mathbb{C} of degree $m_k - \deg P_t, t = 1, \dots, s$.

So from $f : \Omega \rightarrow V$, we have

$$\sum_{t=1}^s b_t(f_0(z), \dots, f_N(z)) P_t(f_0(z), \dots, f_N(z)) = 0.$$

This implies

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - d} \max\{|Q_{i_0} \circ (f(z))|, \dots, |Q_{i_n} \circ (f(z))|\},$$

where c_1 is a positive constant depends only on the coefficients of $b_{k_j}, 0 \leq j \leq n, 0 \leq k \leq N$, thus depends only on the coefficients of $Q_l, 1 \leq l \leq q$. Therefore,

$$(3.18) \quad \|f(z)\|^d \leq c_1 \max\{|Q_{i_0} \circ (f(z))|, \dots, |Q_{i_n} \circ (f(z))|\}.$$

By (3.17) and (3.32), we get

$$\prod_{l=1}^q \frac{\|f(z)\|^d \|Q_l\|}{|Q_l(f(z))|} \leq C \prod_{k=0}^n \frac{\|f(z)\|^d \|Q_{i_k}\|}{|Q_{i_k}(f(z))|},$$

where $C = c_1^{q-n-1} \prod_{l \notin \{i_0, \dots, i_n\}} \|Q_l\|$ and $\|Q_l\|$ is the maximum of the absolute values of the coefficients of Q_l . Thus, we have

$$\begin{aligned} \sum_{l=1}^q m_f(r, D_l) &= \int_0^{2\pi} \sum_{l=1}^q \log \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \sum_{j=1}^M \int_0^{2\pi} \sum_{l=1}^q \log \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d}{|Q(f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log \prod_{l=1}^q \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} \frac{d\theta}{2\pi} + \sum_{j=1}^M \int_0^{2\pi} \log \prod_{l=1}^q \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d}{|Q(f)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi}. \end{aligned}$$

Hence, we get

$$\begin{aligned} \sum_{l=1}^q m_f(r, D_l) &\leq \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \left\{ \log \prod_{k=0}^n \frac{\|f(re^{i\theta})\|^d}{|Q_{i_k}(f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} \\ (3.19) \quad &+ \sum_{j=1}^M \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \left\{ \log \prod_{k=0}^n \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d}{|Q_{i_k}(f)(c_j + \frac{1}{r}e^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1) \end{aligned}$$

By argument as M. Ru [17], we consider the map

$$\psi : x \in V \mapsto [Q_1(x) : \dots : Q_q(x)] \in \mathbb{P}^{q-1}(\mathbb{C}).$$

Put $Y = \psi(V)$. The hypothesis *in general position* implies that ψ is a finite morphism on V and Y is a complex projective subvariety of $\mathbb{P}^{q-1}(\mathbb{C})$ and $\dim Y = n$, $\deg Y := \Delta \leq d^n \deg V$. For each $a = (a_1, \dots, a_q) \in \mathbb{Z}_{\geq 0}^q$, denote by $y^a = y_1^{a_1} \dots y_q^{a_q}$. Let m be a positive integer, we consider the vector space $V_m = \mathbb{C}[y_1, \dots, y_q]_m / (I_Y)_m$, where I_Y is the prime ideal which is defined algebraic variety Y , $(I_Y)_m := \mathbb{C}[y_1, \dots, y_q]_m \cap I_Y$. Fix a basis $\{\phi_0, \dots, \phi_{n_m}\}$ of V_m , where $n_m + 1 = H_Y(m) = \dim V_m$. Set,

$$F = [\phi_0(\psi \circ f) : \dots : \phi_{n_m}(\psi \circ f)] : \Omega \rightarrow \mathbb{P}^{n_m}(\mathbb{C}).$$

Note that, f is algebraically non-degenerate, then F is also. For any $c \in \mathbb{R}_{\geq 0}^q$, the Hilbert function of Y with respect to the weight c is defined by

$$S_Y(m, c) = \max \sum_{i=1}^{H_Y(m)} a_i \cdot c,$$

where the maximum is taken over all sets of monomials $y^{a_1}, \dots, y^{a_{H_Y(m)}}$ whose $y^{a_1} + (I_Y)_m, \dots, y^{a_{H_Y(m)}} + (I_Y)_m$ is a basis of $\mathbb{C}[y_1, \dots, y_q]_m / (I_Y)_m$. For every $z \in \Omega$, denote $c_z = (c_{1,z}, \dots, c_{q,z})$, where $c_{l,z} = \log \frac{\|f(z)\|^d \|Q_l\|}{|Q_l(f(z))|}$, $l = 1, \dots, q$. We see that $c_z \in \mathbb{R}_{\geq 0}^q$, for all $z \in \Omega$. There exists a subset $I_z \subset \{0, \dots, q_m\}$, $q_m = C_{q+m-1}^m - 1$, $|I_z| = n_m + 1 = H_Y(m)$ which $\{y^{a_i} : i \in I_z\}$ is a basis of $\mathbb{C}[y_1, \dots, y_q]_m / (I_Y)_m$ (residue classes modulo $(I_Y)_m$) and

$$S_Y(m, c_z) = \max \sum_{i=1}^{H_Y(m)} a_i \cdot c_z.$$

From two basis $\{y^{a_i} : i \in I_z\}$ and $\{\phi_0, \dots, \phi_{n_m}\}$, there exist the forms independent linearly $\{L_{l,z}, l \in I_z\}$ such that

$$y^{a_l} = L_{l,z}(\phi_0, \dots, \phi_{n_m}).$$

We denote J by the set of indices of the linear forms $L_{l,z}$. We see

$$\log \prod_{i \in J} \frac{1}{|L_i(F)(z)|} = \log \prod_{i \in J} \frac{1}{|Q_1(f)(z)|^{a_{i,1}} \dots |Q_q(f)(z)|^{a_{i,q}}}.$$

This implies

$$\begin{aligned} \max_J \log \prod_{i \in J} \frac{\|F(z)\|}{|L_i(F)(z)|} &\geq S_Y(m, c_z) - dm H_Y(m) \log \|f(z)\| \\ (3.20) \quad &+ (n_m + 1) \log \|F(z)\|. \end{aligned}$$

By Lemma 3, we have

$$(3.21) \quad S_Y(m, c_z) \geq \frac{m H_Y(m)}{(n+1)\Delta} e_Y(c_z) - H_Y(m)(2n+1)\Delta \cdot \max_{1 \leq i \leq q} c_{i,z}.$$

From Lemma 4 and D_1, \dots, D_q are in general position on V , for any $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$, we have

$$(3.22) \quad E_Y(c_z) \geq (c_{i_0,z} + \dots + c_{i_n,z})\Delta.$$

Using the definition of c_z , we obtain

$$(3.23) \quad c_{i_0,z} + \dots + c_{i_n,z} = \log \left(\frac{\|f(z)\|^d \|Q_{i_0}\|}{|Q_{i_0}(f)(z)|} \dots \frac{\|f(z)\|^d \|Q_{i_n}\|}{|Q_{i_n}(f)(z)|} \right).$$

From (3.20) to (3.23), we have

$$\begin{aligned}
& \log \left(\frac{\|f(z)\|^d \|Q_{i_0}\|}{|Q_{i_0}(f)(z)|} \cdots \frac{\|f(z)\|^d \|Q_{i_n}\|}{|Q_{i_n}(f)(z)|} \right) \\
& \leq \frac{(n+1)}{mH_Y(m)} \left(\max_J \log \prod_{l \in J} \frac{\|F(z)\|}{|L_l(F)(z)|} - (n_m + 1) \log \|F(z)\| \right) \\
& \quad + d(n+1) \log \|f(z)\| + \frac{(2n+1)(n+1)\Delta}{m} \max_{1 \leq i \leq q} c_{i,z} \\
& = \frac{(n+1)}{mH_Y(m)} \left(\max_J \log \prod_{l \in J} \frac{\|F(z)\|}{|L_l(F)(z)|} - (n_m + 1) \log \|F(z)\| \right) \\
& \quad + d(n+1) \log \|f(z)\| \\
(3.24) \quad & + \frac{(2n+1)(n+1)\Delta}{m} \left(\max_{1 \leq l \leq q} \log \frac{\|f(z)\|^d \|Q_l\|}{|Q_l(f)(z)|} \right).
\end{aligned}$$

Take the integration of both sides of (3.24), we have

$$\begin{aligned}
& \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \log \left(\frac{\|f(re^{i\theta})\|^d \|Q_{i_0}\|}{|Q_{i_0}(f)(re^{i\theta})|} \cdots \frac{\|f(re^{i\theta})\|^d \|Q_{i_n}\|}{|Q_{i_n}(f)(re^{i\theta})|} \right) \frac{d\theta}{2\pi} \\
& + \sum_{j=1}^M \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \log \left(\frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d \|Q_{i_0}\|}{|Q_{i_0}(f)(c_j + \frac{1}{r}e^{i\theta})|} \cdots \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d \|Q_{i_n}\|}{|Q_{i_n}(f)(c_j + \frac{1}{r}e^{i\theta})|} \right) \frac{d\theta}{2\pi} \\
& \leq \frac{(n+1)}{mH_Y(m)} \left(\int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(re^{i\theta})\|}{|L_l(F)(re^{i\theta})|} \frac{d\theta}{2\pi} \right) \\
& + \sum_{j=1}^M \frac{(n+1)}{mH_Y(m)} \left(\int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(c_j + \frac{1}{r}e^{i\theta})\|}{|L_l(F)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} - (n_m + 1)T_F(r) \right) \\
& + d(n+1)T_f(r) + \frac{(2n+1)(n+1)\Delta}{m} \sum_{1 \leq l \leq q} m_f(r, D_l)
\end{aligned}$$

This implies

$$\begin{aligned}
& \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \log \left(\frac{\|f(re^{i\theta})\|^d}{|Q_{i_0}(f)(re^{i\theta})|} \cdots \frac{\|f(re^{i\theta})\|^d}{|Q_{i_n}(f)(re^{i\theta})|} \right) \frac{d\theta}{2\pi} \\
& + \sum_{j=1}^M \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \log \left(\frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d}{|Q_{i_0}(f)(c_j + \frac{1}{r}e^{i\theta})|} \cdots \frac{\|f(c_j + \frac{1}{r}e^{i\theta})\|^d}{|Q_{i_n}(f)(c_j + \frac{1}{r}e^{i\theta})|} \right) \frac{d\theta}{2\pi} \\
& \leq \frac{(n+1)}{mH_Y(m)} \left(\int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(re^{i\theta})\|}{|L_l(F)(re^{i\theta})|} \frac{d\theta}{2\pi} \right) \\
& + \sum_{j=1}^M \frac{(n+1)}{mH_Y(m)} \left(\int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(c_j + \frac{1}{r}e^{i\theta})\|}{|L_l(F)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} \right) \\
& - \frac{(n+1)}{mH_Y(m)} (n_m + 1)T_F(r) + d(n+1)T_f(r) \\
(3.25) \quad & + \frac{(2n+1)(n+1)\Delta}{m} \sum_{1 \leq l \leq q} m_f(r, D_l) + O(1).
\end{aligned}$$

Next, apply to Lemma 5 for F and collection of hyperplanes $L_l, l \in J$, for every $\varepsilon > 0$ and m is large enough, we obtain

$$\begin{aligned}
& \parallel \frac{(n+1)}{mH_Y(m)} \left(\int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(re^{i\theta})\|}{|L_l(F)(re^{i\theta})|} \frac{d\theta}{2\pi} \right) \\
& + \sum_{j=1}^M \int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(c_j + \frac{1}{r}e^{i\theta})\|}{|L_l(F)(c_j + \frac{1}{r}e^{i\theta})|} \frac{d\theta}{2\pi} - (n_m + 1)T_F(r) \Big) \\
(3.26) \quad & \leq -\frac{n+1}{mH_Y(m)} N_W(r, 0) + \frac{\varepsilon}{3m} T_F(r) + O(\log r + \log T_F(r)),
\end{aligned}$$

where $N_W(r, 0)$ is denoted by the Wronskian of F . Combining (3.19), (3.25) and (3.26), we get

$$\begin{aligned}
& \parallel \sum_{l=1}^q m_f(r, D_l) \leq -\frac{n+1}{mH_Y(m)} N_W(r, 0) + \frac{\varepsilon}{3m} T_F(r) + d(n+1)T_f(r) \\
& + \frac{(2n+1)(n+1)\Delta}{m} \sum_{1 \leq l \leq q} m_f(r, D_l) \\
(3.27) \quad & + O(\log r + \log T_F(r)).
\end{aligned}$$

Using the Theorem 1, we see $T_F(r) \leq dmT_f(r) + O(1)$. Thus, (3.27) implies

$$\begin{aligned}
 \parallel \quad \sum_{l=1}^q d(q - (n+1) - \varepsilon/3)T_f(r) &\leq \sum_{l=1}^q N_f(r, D_l) - \frac{n+1}{mH_Y(m)}N_W(r, 0) \\
 &\quad + \frac{(2n+1)(n+1)\Delta}{m} \sum_{1 \leq l \leq q} m_f(r, D_l) \\
 &\quad + O(\log r + \log T_f(r)).
 \end{aligned}
 \tag{3.28}$$

By an argument method in [3], we conclude

$$\begin{aligned}
 \frac{n+1}{mH_Y(m)} \sum_{l=1}^q N_f(r, D_l) - N_W(r, 0) &\leq \frac{n+1}{mH_Y(m)} \sum_{l=1}^q N_f^{n_m}(r, D_l) \\
 &\quad + (2n+1)\Delta H_Y(m) \sum_{l=1}^q N_f(r, D_l).
 \end{aligned}
 \tag{3.29}$$

Combining (3.28) and (3.29), we have

$$\begin{aligned}
 \parallel \quad d(q - (n+1) - \varepsilon/3)T_f(r) &\leq \sum_{l=1}^q N_f^{n_m}(r, D_l) + \frac{(2n+1)(n+1)\Delta}{m} \sum_{l=1}^q N_f(r, D_l) \\
 &\quad + \frac{(2n+1)(n+1)\Delta}{m} \sum_{l=1}^q m_f(r, D_l) \\
 &\leq \sum_{l=1}^q N_f^{n_m}(r, D_l) + \frac{(2n+1)(n+1)dq\Delta}{m} T_f(r) \\
 &\quad + O(\log r + \log T_f(r)).
 \end{aligned}
 \tag{3.30}$$

We choose the m sufficiently large such that

$$\frac{(2n+1)(n+1)\Delta}{m} < \varepsilon/3.
 \tag{3.31}$$

We may choose $m = 18n^2\Delta I(\varepsilon^{-1})$ for the inequality (3.31), where $I(x) := \min\{k \in \mathbb{N} : k > x\}$ for each positive constant x . Thus, from (3.30) and (3.31), we get the inequality

$$\begin{aligned}
 \parallel \quad d(q(1 - \varepsilon/3) - (n+1) - \varepsilon/3)T_f(r) &\leq \sum_{l=1}^q N_f^{n_m}(r, D_l) \\
 &\quad + O(\log r + \log T_f(r)).
 \end{aligned}$$

By property $\deg Y = \Delta \leq d^n \deg V$, where $d = \text{lcm}\{d_1, \dots, d_q\}$, $\deg Y = n$ and $n_m \leq \Delta C_{m+n}^n$, we have

$$\begin{aligned} n_m &\leq \Delta \frac{(m+1)(m+2)\dots(m+n)}{n!} \\ &< \Delta \left(\frac{m+n}{n}\right)^n \frac{n^n}{n!} \\ &= \Delta \left(1 + \frac{m}{n}\right)^n \frac{n^n}{n!}. \end{aligned}$$

For the choice of m , we have

$$n_m \leq \frac{n^n d^{n^2+n} (19nI(\varepsilon^{-1}))^n (\deg V)^{n+1}}{n!}.$$

□

Proof of Theorem 5. Let $\mathbf{f} = (f_0 : \dots : f_N)$ be a reduced representation of f , where f_0, \dots, f_N are entire functions on Ω and have no common zeros. We consider the function $\phi_i = Q_i \circ \mathbf{f} = Q_i(f_0, \dots, f_N)$, $0 \leq i \leq N$. Let $F = (\phi_0 f_0^n : \dots : \phi_N f_N^n)$. Since the hypersurfaces $\{\mathcal{H}_i^n Q_i = 0\}$, $0 \leq i \leq N$, are located in general position in $\mathbb{P}^N(\mathbb{C})$, then $F : \Omega \rightarrow \mathbb{P}^N(\mathbb{C})$ is a holomorphic curve. Let \mathfrak{H}_i , $0 \leq i \leq N$, be the hypersurface defined by $\{\mathcal{H}_i^n Q_i = 0\}$, $0 \leq i \leq N$. From the hypothesis $\mathfrak{H}_0, \dots, \mathfrak{H}_N$ are in general position, i.e.

$$\text{supp} \mathfrak{H}_0 \cap \dots \cap \text{supp} \mathfrak{H}_N = \emptyset.$$

Thus by Hilbert's Nullstellensatz [19], for any integer k , $0 \leq k \leq N$, there is an integer $m_k > n + d$ such that

$$x_k^{m_k} = \sum_{i=0}^N b_i(x_0, \dots, x_N) \mathcal{H}_i^n(x_0, \dots, x_N) Q_i(x_0, \dots, x_N),$$

where b_0, \dots, b_N are homogeneous forms with coefficients in \mathbb{C} of degree $m_k - (n + d)$. This implies

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k - (n+d)} \max\{|\mathcal{H}_0^n Q_0(f(z))|, \dots, |\mathcal{H}_N^n Q_N(f(z))|\},$$

where c_1 is a positive constant depending only on the coefficients of b_i , $0 \leq i \leq N$, thus depending only on the coefficients of Q_i , $0 \leq i \leq N$. Therefore,

$$(3.32) \quad \|f(z)\|^{n+d} \leq c_1 \max\{|\mathcal{H}_0^n Q_0(f(z))|, \dots, |\mathcal{H}_N^n Q_N(f(z))|\}.$$

From (3.32) and the First Main Theorem, we have

$$\begin{aligned}
 T_F(r) &\geq (n+d)T_f(r) + O(1) \\
 &\geq (n+d-(N+1)d)T_f(r) + \sum_{i=0}^N N_f(r, D_i) + O(1) \\
 (3.33) \quad &= (n-Nd)T_f(r) + \sum_{i=0}^N N_f(r, D_i) + O(1).
 \end{aligned}$$

On the other hand, by applying Theorem 2 to F , and the hyperplanes

$$H_i = \{y_i = 0\}, 0 \leq i \leq N,$$

and

$$H_{N+1} = \{y_0 + \cdots + y_N = 0\}$$

yields

$$(3.34) \quad \|T_F(r) \leq \sum_{i=0}^{N+1} N_F^N(r, H_i) + o(T_f(r)).$$

We have

$$N_F^N(r, H_i) \leq N_f^N(r, D_i) + N^N(r, \frac{1}{f_i^n})$$

for all $i = 0, \dots, N$, where $N^N(r, \frac{1}{g})$ is counting function with level of truncation N of g . Hence

$$\begin{aligned}
 N_F^N(r, H_i) &\leq N_f^N(r, D_i) + N\overline{N}(r, \frac{1}{f_i^n}) \\
 (3.35) \quad &\leq N_f^N(r, D_i) + NT_f(r) + O(1)
 \end{aligned}$$

for all $i = 0, \dots, N$. Also note $N_F^N(r, H_{N+1}) = N_f^N(r, D)$. By combining (3.33) to (3.35), we obtain

$$\begin{aligned}
 \|(n-(d+N+1)N)T_f(r) + \sum_{i=0}^N (N_f(r, D_i) - N_f^N(r, D_i)) \\
 \leq N_f^N(r, D) + o(T_f(r)).
 \end{aligned}$$

□

Proof of Theorem 7. We suppose that $f \not\equiv g$, then there are two numbers $\alpha, \beta \in \{0, \dots, N\}$, $\alpha \neq \beta$ such that $f_\alpha g_\beta \not\equiv f_\beta g_\alpha$. Assume that $z_0 \in \Omega$ is a zero of $P(f)$,

from condition $f(z) = g(z)$ when $z \in f^{-1}(D) \cup g^{-1}(D)$, we get $f(z_0) = g(z_0)$.

This implies z_0 is a zero of $\frac{f_\alpha}{f_\beta} - \frac{g_\alpha}{g_\beta}$. Therefore, we have

$$\begin{aligned} N_f^N(r, D) &\leq NN_f^1(r, D) \leq NN \frac{f_\alpha}{f_\beta} - \frac{g_\alpha}{g_\beta}(r, 0) \\ &\leq N(T_f(r) + T_g(r)) + O(1). \end{aligned}$$

Apply to Theorem 5, we obtain

$$(3.36) \quad \|(n - (d + N + 1)N)T_f(r) \leq N(T_f(r) + T_g(r)) + o(T_f(r)).$$

Similarly, we have

$$(3.37) \quad \|(n - (d + N + 1)N)T_g(r) \leq N(T_f(r) + T_g(r)) + o(T_g(r)).$$

Combining (3.36) and (3.37), we get

$$\|(n - (d + N + 3)N)(T_f(r) + T_g(r)) \leq o(T_f(r)) + o(T_g(r)).$$

This is a contradiction with $n > (d + N + 3)N$. Hence $f \equiv g$. \square

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